Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Linear Algebra and its Applications 365 (2003) 301–309

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

From elementary calculations to Hall polynomials

R. Nörenberg *

Institut fur Experimentelle Mathematik, Universitat Essen, Ellernstrasse 29, 45326 Essen, Germany

Received 22 March 2001; accepted 16 April 2002

Submitted by C.M. Ringel

Abstract

Via the computation of cokernels and using the theorem of Krull, Remak and Schmidt to decompose a given representation, the task of computing Hall polynomials for Dynkin quivers can be reduced to solving (many) systems of linear equations. The resulting method has been applied to obtain Hall polynomials for a quiver of type $\mathbf{E}_8$.

Keywords: Quiver; Representation; Dynkin diagrams; Hall polynomials; Quantized enveloping algebra; Module decomposition

1. Introduction

Hall polynomials for a Dynkin quiver Q, which were shown to exist in [9], were used in [10] as structure coefficients to define a Hall algebra structure on the free $\mathbb{Z}[q]$ module with basis the isomorphism classes of all representations of Q over a (finite) field k. That way a Hall algebra isomorphic to the positive part of the quantised enveloping algebra associated to the underlying Dynkin diagram of Q is obtained. Thus explicit knowledge of the Hall polynomials for Dynkin quivers allows for explicit calculations within quantum groups.

While is is not always easy to actually compute these polynomials, given sufficient computational power, it is possible to reduce their calculation to solving systems of linear equations. That way, the Hall polynomials corresponding to short exact sequences with indecomposable first and middle term have for a quiver of type $\mathbf{E}_8$

* Tel.: +4930-314-25-736; fax: +4930-314-79-706.

0024-3795/03/$ - see front matter © 2002 Published by Elsevier Science Inc.
doi:10.1016/S0024-3795(02)00480-9

been established. With an algorithm presented in [4] from these, all Hall polynomials for a quiver of that type can be derived.

2. Elementary computations for representations of quivers

2.1. Notions and notations

We consider a finite *quiver* Q, i.e. a directed graph with a finite set of vertices Q_0 and a finite set of arrows Q_1. For each arrow $\alpha \in Q_1$ we denote by $s(\alpha)$ its initial and by $t(\alpha)$ its terminal vertex.

A *representation* V of the quiver q over a field k is given by a collection $(V_i)_{i\in Q_0}$ of k-vector spaces associated with the vertices of Q, together with a selection of k-linear maps $V_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ associated to the arrows of Q. Such a representation is also denoted by $V = (V_i, V_\alpha)$. We will only consider *finite-dimensional* representation, i.e. representations (V_i, V_α) for which the vector spaces V_i all are of finite dimension over k.

By the *dimension vector* $\underline{\dim}(V)$ of a representation we will understand the tuple $\underline{\dim}(V) = (\dim_k V_i)_{i\in Q_0}$ of the k-dimensions of the associated vector spaces.

2.2. Homomorphisms

Given two representations V, W of Q, a *homomorphism* $\varphi : V \to W$ of representations is given by a tuple $\varphi = (\varphi_i)_{i\in Q_0}$ of k-linear maps $\varphi_i : V_i \to W_i$ for each vertex of Q and such that for each arrow α of the quiver Q the following diagram commutes:

$$\begin{array}{ccc} V_{s(\alpha)} & \stackrel{\varphi_{s(\alpha)}}{\to} & W_{s(\alpha)} \\ \downarrow V_\alpha & & \downarrow W_\alpha \\ V_{t(\alpha)} & \stackrel{\varphi_{t(\alpha)}}{\to} & W_{t(\alpha)} \end{array}$$

The set of all homomorphisms $\varphi : V \to W$ between two representations V and W of the quiver Q over the field k will be denoted by $\mathrm{Hom}_{kQ}(V, W)$.

Given a homomorphism $\varphi : V \to W$, choosing bases of the vector spaces V_i and W_i, $i \in Q_0$, we can identify the k-linear maps V_α, W_α, $\alpha \in Q_1$, and φ_i, $i \in Q_0$, with their associated matrices. For a tuple $(X_i)_{i\in Q_0}$ of $\dim_k W_i \times \dim_k V_i$ matrices X_i, the defining conditions $W_\alpha X_{s(\alpha)} = X_{t(\alpha)} V_\alpha$ for the tuple to be a homomorphism clearly are linear in the entries of the matrices X_i. Thus for explicitly given representations V and W, the computation of the vector space $\mathrm{Hom}_{kQ}(V, W)$ amounts to the solution of a system of linear equations.

2.3. Kernels and cokernels

Given a homomorphism $\varphi : V \to W$, it is possible to obtain a kernel for φ the following way:

$$\begin{array}{ccccccc} 0 & \rightarrow & K_{s(\alpha)} & \xrightarrow{\kappa_{s(\alpha)}} & V_{s(\alpha)} & \xrightarrow{\varphi_{s(\alpha)}} & W_{s(\alpha)} \\ & & \downarrow K_\alpha & & \downarrow V_\alpha & & \downarrow W_\alpha \\ 0 & \rightarrow & K_{t(\alpha)} & \xrightarrow{\kappa_{t(\alpha)}} & V_{t(\alpha)} & \xrightarrow{\varphi_{t(\alpha)}} & W_{t(\alpha)} \end{array}$$

Fig. 1. Computing kernels.

First, for each vertex $i \in Q_0$, choose a kernel $\kappa : K_i \to V_i$ of the k-linear map φ_i. Now, for each arrow α of the quiver Q we have $\varphi_{t(\alpha)}V_\alpha\kappa_{s(\alpha)} = W_\alpha\varphi_{s(\alpha)}\kappa_{s(\alpha)} = 0$, as φ is a homomorphism of representations and $\kappa_{s(\alpha)}$ a kernel of $\varphi_{s(\alpha)}$. Thus, as $\kappa_{t(\alpha)}$ is a kernel of $\varphi_{t(\alpha)}$, there is a unique k-linear map K_α such that $\kappa_{t(\alpha)}K_\alpha = V_\alpha\kappa_{s(\alpha)}$. That way, we obtain a representation $K = (K_i, K_\alpha)$, and by construction, $\kappa = (\kappa_i)_{i\in Q_0}$ becomes a homomorphism $\kappa : K \to V$ that in fact is a kernel of the homomorphism φ.

As the calculation of the kernel of a linear map is a linear problem and also the defining conditions $\kappa_{t(\alpha)}K_\alpha = V_\alpha\kappa_{s(\alpha)}$ on the linear maps K_α are linear, the computation of the kernel of an explicitly given homomorphism of representations of a quiver is a matter of solving systems of linear equations (Fig. 1). By duality, the same holds for the computation of cokernels.

3. Hall polynomials for Dynkin quivers

From now on, let k be a *finite* field, whose cardinality will be denoted by $|k| = q$, and let Q be a Dynkin quiver of type $\mathbf{A}_n$, $\mathbf{D}_n$, $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$, i.e. the underlying graph Δ of Q is the Dynkin diagram of the corresponding type.

The indecomposable representations of such a quiver were in [5] shown to be finitely many up to isomorphism, with explicit descriptions provided. In fact, the isomorphism classes of indecomposable representations are in bijection to the positive roots of a certain quadratic form $q_Q : \mathbb{Z}^{|Q_0|} \to \mathbb{Z}$, the bijection being given by mapping an indecomposable representation N to its dimension vector $\underline{\dim}\, N$. In terms of the quiver, that quadratic form is given as

$$q_Q(x) = \sum_{i\in Q_0} x_i^2 - \sum_{\alpha\in Q_1} x_{s(\alpha)}x_{t(\alpha)}.$$

Note that this form is independent of the orientations of the edges of Q and thus in fact a property of the underlying Dynkin diagram δ, wherefore we will also write q_Δ instead of q_Q and denote the system of positive roots (i.e. the set of those $x \in \mathbb{Z}^{|Q_0|}$ with $q_\Delta(x) = 1$) by Φ_Δ^+.

Let $\mathscr{I}_Q$ denote a system of representatives of the isomorphismclasses of indecomposable representations of the quiver Q. By the Krull Schmidt theorem, given a representation V there are between unique numbers $\nu_{V,U}$, $U \in \mathscr{I}_Q$ such that $V \cong \coprod_{U\in\mathscr{I}_Q} U^{\nu_{V,U}}$. That way the isomorphismclasses of all finite dimensional representations of the quiver Q are in bijection to the functions on the system of positive roots of q_Δ taking values in the nonnegative integers ($\mathbb{N}_0^{\Phi_\Delta^+}$). This bijection is explicitly given by

$$\Psi : V \mapsto \left(\underline{\dim}\, U \mapsto \nu_{V,U},\, U \in \mathscr{I}_Q\right).$$

The field k being finite, it is possible for given representations L, M, and N to count the number of subrepresentations U of L isomorphic to M with factor L/U isomorphic to N. So let us set

$$F_{N,M}^L = \left|\left\{U \subseteq L \text{ subrepresentation} \mid U \cong M,\, L/U \cong N\right\}\right|.$$

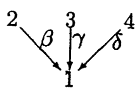
The bijection Ψ between representations the quiver Q and functions on the root system Φ_Δ^+ makes it possible to compare these numbers for different fields. In particular it was shown in [9] that for functions $f, g, h \in \mathbb{N}_0^{\Phi_\Delta^+}$ there is a polynomial $\phi_{h,g}^f \in \mathbb{Z}[x]$ such that for any finite field one has

$$\phi_{h,g}^f(|k|) = F_{\Psi^{-1}(h),\Psi^{-1}(g)}^{\Psi^{-1}(f)}.$$

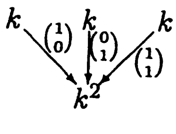
These polynomials $\phi_{h,g}^f$ are called *Hall polynomials*. For given representations L, M, N, we will also write shortly $\phi_{N,M}^L$ for $\phi_{\Psi(N),\Psi(M)}^{\Psi(L)}$.

3.1. An example

We consider the following quiver of type $\mathbf{D}_4$:

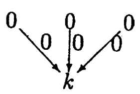
Q: 

Let L be the indecomposable representation visualised by the following picture:

$L =$ 

In this notation, we have written the vector spaces L_i at the corresponding vertices of the quiver Q, and have given explicit matrices for the linear maps L_α, which are written next to their corresponding arrows.

Also, let P be the simple projective module:

$P =$ 

We aim to determine all representations N of Q such that the Hall polynomial $\phi_{N,P}^L$ is nonzero, and to calculate these polynomials.

By definition, those representations N are just the cokernels of monomorphisms $\varphi : R \to L$. The vector spaces P_2, P_3, and P_4 being zero, obviously any homomorphism $\varphi : P \to L$ is just given by the fibre φ_1. Also, any nonzero homomorphism is

Table 1

$\operatorname{Im}\varphi_1$	$\operatorname{coker}\varphi$	$\phi^{L}_{\operatorname{coker}\varphi, P}$
$k\begin{pmatrix}1\\0\end{pmatrix} = \operatorname{Im} L_\beta$	$\begin{tikzcd}[sep=small] k \arrow[dr, "0"'] & 0 \arrow[d, "0"] & 0 \arrow[dl, "0"] \\ & 0 & \end{tikzcd} \amalg \begin{tikzcd}[sep=small] 0 \arrow[dr, "0"'] & k \arrow[d, "1"] & k \arrow[dl, "1"] \\ & k & \end{tikzcd}$	1
$k\begin{pmatrix}0\\1\end{pmatrix} = \operatorname{Im} L_\gamma$	$\begin{tikzcd}[sep=small] 0 \arrow[dr, "0"'] & k \arrow[d, "0"] & 0 \arrow[dl, "0"] \\ & 0 & \end{tikzcd} \amalg \begin{tikzcd}[sep=small] k \arrow[dr, "1"'] & 0 \arrow[d, "0"] & k \arrow[dl, "1"] \\ & k & \end{tikzcd}$	1
$k\begin{pmatrix}1\\1\end{pmatrix} = \operatorname{Im} L_\delta$	$\begin{tikzcd}[sep=small] 0 \arrow[dr, "0"'] & 0 \arrow[d, "0"] & k \arrow[dl, "0"] \\ & 0 & \end{tikzcd} \amalg \begin{tikzcd}[sep=small] k \arrow[dr, "1"'] & k \arrow[d, "1"] & 0 \arrow[dl, "0"] \\ & k & \end{tikzcd}$	1
Any other	$\begin{tikzcd}[sep=small] k \arrow[dr, "1"'] & k \arrow[d, "1"] & k \arrow[dl, "1"] \\ & k & \end{tikzcd}$	$q-2$

a monomorphism. Thus the submodules U of L isomorphic to L, being the monomorphic images of P in L are given by the monomorphic images of $P_1 = k$ in $L_1 = k^2$, i.e. the one dimensional subspaces of the two dimensional vector space $L_1 = k^2$.

Now, given a monomorphism $\varphi : R \to L$, there are essentially two possible situations for its image:

If the image $\operatorname{Im}\varphi_1$ of φ_1 coincides with one of the images of the linear maps L_β, L_γ, or L_δ, then the cokernel decomposes, for each of the corresponding three cases into two different summands. Explicit descriptions of these decompositions are given in Table 1.

On the other hand, if $\operatorname{Im}\varphi$ coincides with neither $\operatorname{Im} L_\beta$, $\operatorname{Im} L_\gamma$ nor $\operatorname{Im} L_\delta$, the cokernel is always the unique indecomposable representation of Q with dimension vector $(1, 1, 1, 1)$.

This, each of the tree occurring decomposing cokernels correspond to a single one dimensional subspace of L_1 whereas the indecomposable cokernel corresponds to all remaining one dimensional subspaces, of which there are $q + 1 - 3 = q - 2$ (recalling the number of one dimensional subspaces of an n dimensional vector space to be $1 + q + \cdots + q^{n-1}$).

4. Hall polynomials for $\mathbf{E}_8$

Our aim is to have the Hall polynomials $\phi^{L}_{N,M}$ for indecomposable representations M and L computed for a quiver of type $\mathbf{E}_8$. This in fact includes the cases of $\mathbf{E}_6$ and $\mathbf{E}_7$, which have been done by hand by C.M. Ringel.[1]

[1] Unpublished unfortunately.

For this let Q be the following quiver:

$$
\begin{array}{ccccc}
 & & & & 8 \\
 & & & & \downarrow \\
 & & & & 7 \\
 & & & & \downarrow \\
Q: & & 4 & & 6 \\
 & & \downarrow & & \downarrow \\
 & 2 & 3 & & 5 \\
 & \searrow & \downarrow & \swarrow & \\
 & & 1 & &
\end{array}
$$

This means, we work with just one of many possible orientations of the arrows. On how to change between different orientations, we refer to [11]. Also by [11] we may assume M to be projective. As for larger representations L the computation of Hall polynomials becomes a rather lengthy and tedious business, we would wish to pass this on to a computer, having it perform the following task:

> For L indecomposable and M indecomposable projective compute for all relevant representations N the numbers $F^L_{N,M}$ for sufficiently many finite fields of different cardinality, to obtain the polynomials $\phi^L_{N,M}$ by interpolation.

From this, two questions arise:

1. How many are sufficiently many fields?
2. What are the relevant representations N?

4.1. *Projective representations*

To address the questions just stated, we first take a look at the indecomposable projective representations of Q. These are indexed by the vertices of Q and can be obtained in the following way:

Fix a vertex $i \in Q_0$. For $j \in Q_0$ set $P_j = k$ if there is a chain $\alpha_1, \alpha_2, \ldots, \alpha_n$ of arrows in Q with $n \geqslant 0$, such that $s(\alpha_1) = i$, $t(\alpha_n) = j$, and $t(\alpha_l) = s(\alpha_{l+1})$ for $l \in \{1, \ldots, n-1\}$. Otherwise let $P_j = 0$. Now for an arrow $\alpha \in Q_1$, let $P_\alpha = 1$ (i.e. the identity map on k) if both $P_{s(\alpha)} = k$ and $P_{t(\alpha)} = k$. Otherwise set $P_\alpha = 0$. This way, we obtain a representation $P(i) := P = (P_i, P_\alpha)$ of Q which is indecomposable and projective.

4.1.1. *Homomorphisms from indecomposable projective representations*

Let $P = P(i)$ be an indecomposable projective representation of Q and let $\varphi \in \mathrm{Hom}_{kQ}(P, V)$, with V an arbitrary representation. We consider a vertex $j \in Q_0 \backslash \{i\}$ and assume $P_j = k$ (as otherwise $\varphi = 0$). By definition then there is a chain α_1,

$\alpha_2, \ldots, \alpha_n \in Q_0$ with $s(\alpha_1) = i$, $t(\alpha_n) = j$, and $t(\alpha_l) = s(\alpha_{l+1})$ for $l \in \{1, \ldots, n-1\}$. As φ is a homomorphism we have $\varphi_j = \varphi_j \cdot 1 = V_{\alpha_n}\varphi_{s(\alpha_n)}$, and by induction along the chain thus: $\varphi_j = \varphi_{s(\alpha_n)}\varphi_{s(\alpha_{n-1})} \cdots \varphi_{s(\alpha_1)}\varphi_i$. So φ_j is uniquely determined by φ_i, and thus so is φ. We obtain:

Lemma. *Let $P = P(i)$ and V be as stated above. Then $\Xi : \mathrm{Hom}_{kQ}(P, V) \to \mathrm{Hom}_k(P_i, V_i)$, $\varphi \mapsto \varphi_i$ is an isomorphism of vectorspaces.*

Let us stress, that the preceding arguments give explicit construction for the inverse of Ξ. Also it should be mentioned, that these constructions for projectives and their homomorphism hold more generally whenever the underlying graph of the quiver is a tree.

4.2. Degree bounds

The orientation of the arrows of our quiver Q has been chosen such that for any indecomposable representation V each of the k-linear maps V_α, $\alpha \in Q_1$, is either zero or a monomorphism. Thus, fixing an indecomposable representation L of Q and an indecomposable projective representation $P = P(i)$, either there is no monomorphism $\varphi : P \to L$ (which is the case if and only of there is an arrow α with $P_\alpha \neq 0$ but $L_\alpha = 0$), or every nonzero homomorphism $\varphi : P \to L$ is a monomorphism.

In the first case, obviously $\phi^L_{N,P} = 0$ for all representations N.

In the latter case, by the Lemma, the monomorphic images of P in L correspond to the monomorphic images of $k = P_i$ in L_i. Thus for a field k of cardinality q we get $\sum_N F^L_{N,P} = q^{\dim L_i - 1} + \cdots + q + 1$, and therefore $\sum_N \phi^L_{N,P} = x^{\dim L_i - 1} + \cdots + x + 1$.

Therefore for an arbitrary representation N the degree of $\phi^L_{N,P}$ is bounded by $\dim L_i - 1$ and thus $\dim L_i$ fields of different cardinality are sufficient to calculate $\phi^L_{N,P}$ by interpolation. It addition it is known that the maximal dimension $\dim L_i$ that can occur for an indecomposable representation of Q is 6.

4.3. Computing $F^L_{N,P}$

Let again $P = P(i)$ be indecomposable projective and L be indecomposable. We may assume, that there is at least one subrepresentation of L isomorphic to P. As we have seen above, in this case the subrepresentations U of L isomorphic to P correspond to the k-linear subspaces of L_i isomorphic to $P_i = k$. It is an easy task, to obtain for every of these an explicit embedding $\varphi_i : k \to L_i$. By the explicit construction of the inverse of the isomorphism Ξ above, we obtain from these explicit embeddings $\varphi : P \to L$. For any of these we can compute the cokernel as described in Section 2.3. What remains to do, in order to obtain the numbers $F^L_{N,P}$ is to count how often a specific cokernel N arises during this procedure.

However, although the cokernels are computed explicitly, this explicit representation by matrices is far from being unique. Thus the the question remains, how to identify such a cokernel N. This can be done by explicit decomposition.

4.4. Decomposition

The theorem of Krull, Remak and Schmidt not only assures the uniqueness of the decomposition of a representation of Q into indecomposable summands, its proof also provides the means to calculate these summands.

For this, let V by an arbitrary representation and M be indecomposable. If for bases $\varphi_1, \ldots, \varphi_n$ of $\mathrm{Hom}_{kQ}(M, V)$ and $\psi_1, \ldots, \psi_n$ of $\mathrm{Hom}_{kQ}(V, M)$ all possible compositions $\psi_j \varphi_i$ are zero, then $\mathrm{Hom}_{kQ}(V, M)\mathrm{Hom}_{kQ}(M, V) = 0$ and M cannot be a direct summand of V. On the other hand, it is known, that for the indecomposable representations of Q the endomorphism ring $\mathrm{Hom}_{kQ}(M, M)$ is isomorphic to the field k. Therefore if the composition $\psi_j \varphi_i$ is nonzero for some i and j, then it is an automorphism and thus splits, wherefore M is a direct summand of V. As Q has finitely many indecomposable representations up to isomorphism (the number of ismoprhismclasses of these in fact is exactly 120), finding a direct summand M is a finite task, and a total decomposition of V can be accomplished by iterating the described procedure for the factor V/M.

4.5. Results

An implementation of the described method was used on all pairs (P, L) where P is an indecomposable projective representation and L is indecomposable, up to isomorphism. One result of these computations is the following:

Theorem. *Let Q be a quiver of type $\mathbf{E}_8$. As Hall polynomials $\phi^L_{N,M}$ for M and L indecomposable, exactly the following polynomials occur*:

0	$x^3 - 3x^2 + 4x - 2$
1	$x^3 - 3x^2 + 5x - 4$
$x - 2$	$x^3 - 2x^2 + 2x - 1$
$x - 1$	$x^4 - 6x^3 + 16x^2 - 22x + 12$
$x^2 - 4x + 4$	$x^4 - 5x^3 + 10x^2 - 11x + 6$
$x^2 - 3x + 2$	$x^4 - 5x^3 + 10x^2 - 10x + 4$
$x^2 - 3x + 3$	$x^4 - 5x^3 + 10x^2 - 10x + 5$
$x^2 - 2x + 1$	$x^4 - 5x^3 + 11x^2 - 14x + 8$
$x^2 - x$	$x^4 - 5x^3 + 11x^2 - 13x + 7$
$x^3 - 6x^2 + 12x - 8$	$x^4 - 4x^3 + 6x^2 - 5x + 2$
$x^3 - 5x^2 + 8x - 4$	$x^4 - 4x^3 + 6x^2 - 4x + 1$
$x^3 - 5x^2 + 9x - 6$	$x^4 - 4x^3 + 7x^2 - 9x + 6$
$x^3 - 5x^2 + 10x - 7$	$x^4 - 4x^3 + 7x^2 - 8x + 5$

$x^3 - 4x^2 + 5x - 2$	$x^4 - 4x^3 + 7x^2 - 6x + 2$
$x^3 - 4x^2 + 6x - 4$	$x^4 - 4x^3 + 8x^2 - 10x + 6$
$x^3 - 4x^2 + 6x - 3$	$x^5 - 6x^4 + 15x^3 - 23x^2 + 25x - 13$
$x^3 - 4x^2 + 7x - 5$	$x^5 - 5x^4 + 10x^3 - 13x^2 + 14x - 8$
$x^3 - 4x^2 + 7x - 4$	$x^5 - 5x^4 + 10x^3 - 13x^2 + 15x - 9$
$x^3 - 4x^2 + 8x - 6$	$x^5 - 5x^4 + 10x^3 - 12x^2 + 11x - 6$
$x^3 - 3x^2 + 2x$	$x^5 - 5x^4 + 10x^3 - 12x^2 + 12x - 7$
$x^3 - 3x^2 + 3x - 1$	

The results in detail, as which representations lead to which polynomials, are available as rather independent part of the computer algebra system crep and can be obtained via the webpage `www.mathematik.uni-bielefeld.de/birep/crep/ftpindex.html` or from the directory `pub/math/f-d-alg/crep` on the ftp server `ftp.uni-bielefeld.de`.

References

[1] K. Bongartz, Algebras and quadratic forms, J. London Math. Soc. 28 (1983) 461–469.

[2] L.M. Butler, A.W. Hales, Nonnegative Hall polynomials, J. Algebraic Combin. 2 (1993) 125–135.

[3] P. Dräxler, R. Nörenberg, The CREP manual, Part IV, vol. 00-005, Ergänzungsreihe, SFB 343, University of Bielefeld.

[4] D. Guhe, C.M. Ringel, Computations of Hall Polynomials, preprint.

[5] P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972) 71–103.

[6] Ph. Hall, The algebra of partitions, in: Proc. 4th Canad. Math. Congress, Bannff (1957), University of Toronto Press, 1959.

[7] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., The Claredon Press, Oxford University Press, New York, 1995.

[8] F. Miller Maley, The Hall polynomial revisited, J. Algebra 184 (1996) 363–371.

[9] C.M. Ringel, Hall algebras, in: Topics in Algebra, Part 1, Banach Center Publ. 26, Part 1, PWN, Warsaw, 1990.

[10] C.M. Ringel, The Hall algebra approach to quantum groups, Aportaciones Matemáticas Comunicaciones 15 (1995) 85–114.

[11] C.M. Ringel, Hall polynomials for the representation-finite hereditary algebras, Adv. Math. 84 (1990) 137–178.

[12] D. Simson, Kołczany, diagramy Dynkina, oswojoność i dzikość w teorii reprezentacji algebr, Wiadom. Mat. 33 (1997) 1–46.